J. Fluid Mech. (1971), *vol.* 49, *part* 1, *pp.* 113–131
Printed in Great Britain

The reflexion of internal/inertial waves from bumpy surfaces. Part 2. Split reflexion and diffraction

By P. G. BAINES
Department of Earth and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 24 November 1970)

This paper considers the linear inviscid reflexion of internal/inertial waves from smooth bumpy surfaces where a characteristic (or ray) is tangent to the surface at some point. There are two principal cases. When a characteristic associated with the incident wave is tangent to the surface we have diffraction; when the tangential characteristic is associated with a reflected wave we have split reflexion, a phenomenon which has no counterpart in classical non-dispersive wave theory. In both these cases the problem of determining the wave field may be reduced to a set of coupled integral equations with two unknown functions. These equations are solved for the simplest topography for each case, and the properties of the wave fields for more general topographies are discussed. For both split reflexion and diffraction, the fluid velocity has an inverse-square-root singularity on the tangential characteristic, and the energy density has a corresponding logarithmic singularity. The diffracted wave field penetrates into the shadow region a distance which is of the order of the incident wavelength. Possibilities for instability and mixing are discussed.

1. Introduction and summary

The reflexion of internal and/or inertial waves from a smooth bumpy surface has been discussed in Baines (1971) (hereafter referred to as I) for the case where the wave characteristics (i.e. directions of energy flux) are nowhere tangent to the surface. This paper considers the consequences when a wave characteristic is tangent to the surface at some point ('steep bump' topography, in the terminology of I).

The analysis is two-dimensional, linear and inviscid, and is based on a radiation condition which has the form

$$F(\zeta) = \pm\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{F(\zeta')\,d\zeta'}{\zeta'-\zeta}, \tag{1.1}$$

for any wave field $F(\zeta)\,e^{-i\omega t}$, where ζ is a characteristic variable. Equation (1.1) is simply a statement that the Fourier transform of $F(\zeta)$ vanishes for negative (or positive, depending on appropriate sign) values of its wave-number argument. Not all the work of previous authors satisfies equation (1.1), and for a discussion of these the reader is referred to I. This equation has also arisen in other contexts of wave propagation but with ζ as a frequency and associated with the 'arrow of time' (Rosenfeld 1961).

There are two cases: when the tangential wave characteristic is associated with the incident wave there is a region of the surface which is not 'lit' by the latter (omitting the case of a point of inflexion), and we have diffraction. When the tangential characteristic is associated with a reflected wave we have a new phenomenon termed 'split reflexion'. The mathematical formulation for the determination of the wave field for surfaces which have one of these singular points is given in §3. The analysis is complicated because the equation for the surface is not a single-valued function in terms of the characteristic co-ordinates, and there are three wave-fields to determine: the back-reflected wave, a reflected wave and a second reflected or diffracted wave. In either case the problem may be reduced to a pair of coupled integral equations, one singular and the other non-singular, for two unknown functions. Such equations are discussed by Muskhelishvili (1946). These equations have been solved for the simplest cases, and the properties of the solutions for more general cases are also discussed in §§4 and 5. The results obtained are summarized below.

For split reflexion (see figure 1) the velocity near the tangential characteristic $\eta = 0$ is proportional to $\epsilon k_1(R/|\eta|)^{\frac{1}{2}}$ where η is the characteristic co-ordinate measured perpendicular to it, ϵk_1 is the amplitude of the velocity of the incident wave and R is the radius of curvature of the surface at the tangent point. This inverse-square-root singularity in the velocity will be present whenever a radius of curvature exists at the tangent point, and it clearly becomes stronger with increasing R.

For the case of diffraction (see figure 2) the velocity is also singular on the tangential characteristic $\xi = 0$ and in its neighbourhood has the form

$$\epsilon k_1(k_1/|\xi|)^{\frac{1}{2}}\, e^{-i\omega t},$$

so that the strength of the singularity is independent of the radius of curvature (k_1 is the wave-number of the incident wave). For each of the split reflexion and diffraction cases, a solution is found for a particularly simple surface shape. Solutions for more general topographic shapes will be asymptotic to these solutions in the limit of short incident wavelength, i.e. $k_1 R \gg 1$. For the simplest surface shape in the diffraction case (and therefore for general shapes in the limit $k_1 R$ large), the diffracted wave penetrates the shadow region a distance which is of the order of one wavelength of the incident wave.

It is interesting to compare the above results for diffraction with those of Hurley (1970), who has obtained solutions for a sharp-angled wedge by a Green's function method. Hurley's solutions are singular on all characteristics passing through the vertex, and this singularity in the velocity is nearly $O(1/|\eta|^{\frac{1}{3}})$ for one range of wedge angles and nearly $O(1/|\eta|^{\frac{1}{2}})$ for the other.

In §6 it is shown in general that the stream functions for reflected and diffracted wave fields have the asymptotic form $(1/\zeta)\, e^{-i\omega t}$ ($|\zeta|$large) in the directions perpendicular to the lines of constant ζ where ζ is again the appropriate characteristic variable, provided that the surface asymptotes to planes away from the bumpy region.

The ubiquity of the above-mentioned singularity raises questions as to the relevance of the linear inviscid theory for split reflexion and diffraction cases.

It is clear that such topographic features tend to cause energy density to accumulate, as the energy of the incident wave is converted into components of higher wave-number and consequent slower group velocity. Such a concentration of energy is likely to enhance non-linear effects, and some experiments by Cacchione (1970) and others by the author show this. Viscous effects will also be significant. Reflexion in a viscous fluid from a plane boundary was first considered by Phillips (1963) who showed that two boundary layers of thickness $O(\nu/\omega)^{\frac{1}{2}}$ arise, where ν is the kinematic viscosity. If the slope of the reflected characteristics is very close to that of the boundary, however, the reflected wave is trapped in a boundary layer of order $(\nu/k_1\omega)^{\frac{1}{3}}$ (H. P. Greenspan, private communication), in which the shears are very large. For the cases of split-reflexion and diffraction considered in the present paper, instability and subsequent mixing of the stratified fluid is quite plausible if the incident energy flux is sufficiently large. This criterion need not be very demanding, and the phenomenon could well have geophysical applications. However, if the incident wave is below some threshold in amplitude the linear theory should be useful.

2. Basic equations

We consider the motion of an incompressible inviscid rotating stratified fluid, and take Cartesian axes x, y, z, z increasing vertically upward, with corresponding velocity components u, v, w. We take the axis of rotation to be vertical, and the linearized equations of motion in the rotating frame are

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{f}\times\mathbf{u} = -\frac{1}{\rho_0(z)}\nabla p-\frac{\rho g\hat{\mathbf{z}}}{\rho_0(z)},$$

$$\nabla.\mathbf{u} = 0, \tag{2.1}$$

$$\frac{\partial\rho}{\partial t}+w\frac{d\rho_0}{dz} = 0,$$

where $\rho_0(z)$ is the equilibrium density, p and ρ are the perturbation pressure and density respectively, $\hat{\mathbf{z}}$ is the unit vector in the direction of z increasing, t is the time variable, u is the fluid velocity, g the acceleration due to gravity, and $\mathbf{f} = f\hat{\mathbf{z}} = 2\mathbf{\Omega}$ where $\mathbf{\Omega}$ is the angular velocity of the system. We next assume that the bottom topography and the incident wave motion are independent of the y co-ordinate, so that we may define a stream function $\psi(x, z, t)$ by the equations

$$u = -\partial\psi/\partial z, \quad w = \partial\psi/\partial x. \tag{2.2}$$

Equations (2.1) then yield the equation for ψ

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi+N^2\psi_{xx}+f^2\psi_{zz} = 0, \tag{2.3}$$

where N is the Brunt–Väisälä frequency defined by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}. \tag{2.4}$$

If we further assume that all the fluid motion has the time dependence $e^{-i\omega t}$, then writing

$$\psi = \hat{\psi}(x,z)\, e^{-i\omega t}, \tag{2.5}$$

we obtain

$$\hat{\psi}_{xx} - c^2\, \hat{\psi}_{zz} = 0, \quad c^2 = (\omega^2 - f^2)/(N^2 - \omega^2), \tag{2.6}$$

where the suffices denote derivatives. In order to have internal and/or inertial waves we require $c^2 > 0$, and for the sake of definiteness we will take

$$0 < f < \omega < N,$$

which is the case of greatest relevance for the ocean. We also assume that N^2 is constant. The conclusions of the following theory will still be valid in cases where $N^2(z)$ is not constant, however, provided only that N^2 be effectively constant in the regions of the fluid near the bottom topography. With N^2 constant, c^2 is constant and equation (2.6) has the general solution

$$\hat{\psi} = f(\xi) + g(\eta), \tag{2.7}$$

where f and g are arbitrary complex-valued functions of the real characteristic variables $\xi = z + cx$, $\eta = z - cx$.

We consider fluids of effectively infinite depth with a bottom surface or topography which has the equation

$$z = h(x), \tag{2.8}$$

and assume that this surface has a radius of curvature at each point. In terms of the characteristic variables this equation may be written

$$\xi = -K(\eta), \quad \eta = -H(\xi), \tag{2.9}$$

and for the bottom surfaces to be discussed one of these relations will be double-valued. The boundary condition to be satisfied on this surface is

$$\psi = 0. \tag{2.10}$$

An analytical form for the radiation condition appropriate to internal and/or inertial waves emanating from a source with a given frequency has been derived in I. For a wave field which is a function of one characteristic variable, e.g.

$$\psi = F(\eta)\, e^{-i\omega t}, \tag{2.11}$$

the necessary and sufficient condition for it to be composed of plane waves whose phase propagation (and hence, associated energy flux) is in one direction only is

$$F(\eta) = \pm \mathrm{P}\int_{-\infty}^{\infty} \frac{F(\eta')\, d\eta'}{\eta' - \eta}, \tag{2.12}$$

where P denotes a principal value integral, and the sign depends on the direction of the energy flux. For any particular case, the relevant sign may be determined by considering a single plane wave.

3. Formulation of the reflexion and diffraction problems

We consider the reflexion of a plane wave from a rigid surface where the wave characteristics are such that at one point one of them is tangent to the surface. There are two possible cases, and these will be considered separately.

3.1. *The split reflexion problem*

We consider a plane wave $\epsilon e^{i(k_1 \xi - \omega t)}$ incident on topography as shown in figure 1, with the origin at the point where a characteristic is tangent to the surface. We denote the reflected wave field by a superposition of

$$\left.\begin{aligned} \psi_R &= \epsilon F_2(\xi)\, e^{-i\omega t}, && \text{the back-reflected wave, as in I,} \\ \psi_{Tr} &= -\epsilon F_1(\eta)\, e^{-i\omega t}, \xi > 0, && \text{the wave reflected to the right,} \\ \psi_{TL} &= -\epsilon F_3(\eta)\, e^{-i\omega t}, \xi < 0, && \text{the wave reflected to the left.} \end{aligned}\right\} \tag{3.1}$$

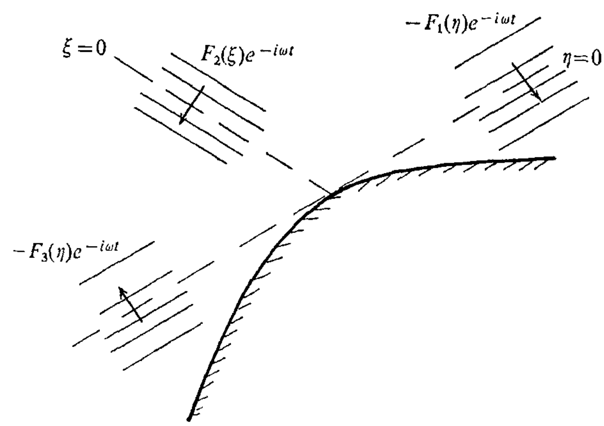

FIGURE 1. Split reflexion topography with reflected wave fields. The latter are shown schematically only, and the incident wave is omitted.

The condition that ψ must vanish on the boundary then yields

$$\left.\begin{aligned} F_1(\eta) &= e^{ik_1\xi} + F_2(\xi), \quad \text{on} \quad \eta = -H(\xi) \quad (\xi > 0), \\ F_3(\eta) &= e^{ik_1\xi} + F_2(\xi), \quad \text{on} \quad \eta = -H(\xi) \quad (\xi < 0), \end{aligned}\right\} \quad (\eta < 0), \tag{3.2}$$

and we also have
$$F_1(\eta) = F_3(\eta) \quad (\eta > 0). \tag{3.3}$$

We assume that at a large distance from the origin in each direction the surface asymptotes to a flat plane, and that there are no other grazing characteristics. The appropriate radiation conditions for the above functions then are

$$F_2(\xi) = \frac{i}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi}, \tag{3.4}$$

$$F_1(\eta) = \frac{i}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{F_1(\eta')\, d\eta'}{\eta' - \eta}, \tag{3.5}$$

$$F_3(\eta) = -\frac{i}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{F_3(\eta')\, d\eta'}{\eta' - \eta}. \tag{3.6}$$

Equations (3.2)–(3.6) then constitute the mathematical problem to be solved, given the form of the surface. The latter may be written

$$\left.\begin{aligned} \eta &= -H_R(\xi), \quad \xi = -K_R(\eta), \quad \text{for} \quad \xi > 0, \\ \eta &= -H_L(\xi), \quad \xi = -K_L(\eta), \quad \text{for} \quad \xi < 0, \end{aligned}\right\} \tag{3.7}$$

and each of these functions is monotonic and single-valued.

We first consider the nature of the function $H(\xi)$ near the origin. We denote the radius of curvature of the surface at the origin by R. Then simple geometric considerations yield that, for $|\xi/R| < 1$,

$$H(\xi) = \frac{(1+c^2)^{\frac{3}{2}}}{8c^2R}\xi^2(1+O(\xi/R)), \tag{3.8}$$

and similarly
$$K(\eta) = -\operatorname{sgn}\xi\frac{2^{\frac{3}{2}}cR^{\frac{1}{2}}}{(1+c^2)^{\frac{3}{4}}}|\eta|^{\frac{1}{2}}(1+O(\eta/R)^{\frac{1}{2}}), \tag{3.9}$$

where $|\eta|$ denotes the modulus of η and $\operatorname{sgn}\xi$ the sign of ξ. Hence near the origin the derivative of $K(\eta)$ necessarily has the form

$$K'(\eta) \sim \operatorname{sgn}\xi\frac{2^{\frac{1}{2}}cR^{\frac{1}{2}}}{(1+c^2)^{\frac{3}{4}}|\eta|^{\frac{1}{2}}}, \tag{3.10}$$

for any surface satisfying the above conditions.

From equations (3.7) we next define the function $\phi(\xi)$ by the following:

$$\begin{aligned} \xi > 0&: H_L(\phi(\xi)) = H_R(\xi),\\ \xi < 0&: H_R(\phi(\xi)) = H_L(\xi), \end{aligned} \tag{3.11}$$

so that for any point (ξ, η) on the surface, the η characteristic will intersect the surface at the point $(\phi(\xi), \eta)$. It is readily established from the foregoing equations that $\phi(\xi)$ has the following properties:

$$\left.\begin{aligned} &\text{(i)}\ \ \phi(0) = 0, \quad \phi'(0) = -1;\\ &\text{(ii)}\ \ \phi(\phi(\xi)) = \xi;\\ &\text{(iii)}\ \ \phi'(\xi) < 0 \quad \text{for all} \quad \xi. \end{aligned}\right\} \tag{3.12}$$

$\phi(\xi)$ represents the degree of asymmetry of the surface about the line $\xi = 0$, such that if the surface is symmetric (in η) about this line we have

$$\phi(\xi) = -\xi, \tag{3.13}$$

for all ξ.

We now obtain a set of equations for $F_2(\xi)$, and to this end define

$$\left.\begin{aligned} F_+(\xi) &= \tfrac{1}{2}[F_2(\xi)+F_2(\phi(\xi))],\\ F_-(\xi) &= \tfrac{1}{2}[F_2(\xi)-F_2(\phi(\xi))], \end{aligned}\right\} \tag{3.14}$$

so that
$$F_2(\xi) = F_+(\xi)+F_-(\xi), \quad F_2(\phi(\xi)) = F_+(\xi)-F_-(\xi). \tag{3.15}$$

Equations (3.2) and (3.5) yield

$$F_1(\eta) = \frac{i}{\pi}\int_0^\infty \frac{F_1(\eta')\,d\eta'}{\eta'-\eta}+\frac{i}{\pi}\int_{-\infty}^0 \frac{(F_2(\xi')+e^{ik_1\xi'})\,d\eta'}{\eta'-\eta} \quad (-\infty < \eta < \infty), \tag{3.16}$$

where $\xi' = -K_R(\eta')$, and so

$$F_1(\eta) = \frac{i}{\pi}\int_0^\infty \frac{F_1(\eta')\,d\eta'}{\eta'-\eta}-\frac{i}{\pi}\int_0^\infty \frac{(F_2(\xi')+e^{ik_1\xi'})\,H_R'(\xi')\,d\xi'}{H_R(\xi')+\eta}, \tag{3.17}$$

for all η. Similarly, (3.2) and (3.6) give

$$F_3(\eta) = -\frac{i}{\pi}\int_0^\infty \frac{F_3(\eta')\,d\eta'}{\eta'-\eta} - \frac{i}{\pi}\int_{-\infty}^0 \frac{(F_2(\xi')+e^{ik_1\xi'})\,d\eta'}{\eta'-\eta} \quad (-\infty < \eta < \infty), \tag{3.18}$$

where now $\xi' = -K_L(\eta')$, and so

$$F_3(\eta) = -\frac{i}{\pi}\int_0^\infty \frac{F_1(\eta')\,d\eta'}{\eta'-\eta} - \frac{i}{\pi}\int_{-\infty}^0 \frac{(F_2(\xi')+e^{ik_1\xi'})\,H_L'(\xi')\,d\xi'}{H_L(\xi')+\eta} \quad (-\infty < \eta < \infty), \tag{3.19}$$

utilizing (3.3). The integrals are to be construed as Cauchy principal value integrals where appropriate. Writing $\xi' = \phi(\xi'')$ in the second integral of equation (3.19), and changing the variable of integration back to ξ' gives

$$F_3(\eta) = -\frac{i}{\pi}\int_0^\infty \frac{F_1(\eta')\,d\eta'}{\eta'-\eta} + \frac{i}{\pi}\int_0^\infty \frac{(F_2(\phi(\xi'))+e^{ik_1\phi(\xi')})\,H_R'(\xi')\,d\xi'}{H_R(\xi')+\eta}, \tag{3.20}$$

for all η, utilizing the properties of $\phi(\xi)$ as given in equations (3.11), (3.12) and writing ξ' for ξ''. Adding equations (3.17) and (3.20) then yields

$$F_1(\eta)+F_3(\eta) = -\frac{i}{\pi}\int_0^\infty \frac{[F_2(\xi')-F_2(\phi(\xi'))+e^{ik_1\xi'}-e^{ik_1\phi(\xi')}]\,H_R'(\xi')\,d\xi'}{H_R(\xi')+\eta} \quad (0 < \eta < \infty). \tag{3.21}$$

Taking $\eta = -H_R(\xi)$, equations (3.2) give

$$F_2(\xi)+F_2(\phi(\xi)) = F_1(\eta)+F_3(\eta)-(e^{ik_1\xi}+e^{ik_1\phi(\xi)}) \quad (\xi > 0), \tag{3.22}$$

and from equations (3.14), (3.21) we finally obtain

$$F_+(\xi) = A(\xi) - \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_-(\xi')\,H_R'(\xi')\,d\xi'}{H_R(\xi')-H_R(\xi)} \quad (0 < \xi < \infty), \tag{3.23}$$

where $$A(\xi) = -\tfrac{1}{2}(e^{ik_1\xi}+e^{ik_1\phi(\xi)}) - \frac{i}{2\pi}\mathrm{P}\int_0^\infty \frac{(e^{ik_1\xi'}-e^{ik_1\phi(\xi')})\,H_R'(\xi')\,d\xi'}{H_R(\xi')-H_R(\xi)}. \tag{3.24}$$

Two other relations between $F_+(\xi)$ and $F_-(\xi)$ may be obtained from equation (3.4). The latter may be written

$$F_2(\xi) = \frac{i}{\pi}\int_0^\infty \frac{F_2(\xi')\,d\xi'}{\xi'-\xi} - \frac{i}{\pi}\int_0^\infty \frac{F_2(\phi(\xi'))\,\phi'(\xi')\,d\xi'}{\phi(\xi')-\xi}, \tag{3.25}$$

and using equations (3.14) we obtain

$$\begin{aligned} F_+(\xi) = {} & \frac{i}{2\pi}\int_0^\infty F_+(\xi')\left[\frac{1}{\xi'-\xi} - \frac{\phi'(\xi')}{\phi(\xi')-\phi(\xi)} + \frac{1}{\xi'-\phi(\xi)} - \frac{\phi(\xi')}{\phi(\xi')-\xi}\right]d\xi' \\ & + \frac{i}{2\pi}\mathrm{P}\int_0^\infty F_-(\xi')\left[\;\ldots\; + \;\ldots\; + \;\ldots\; + \;\ldots\;\right]d\xi', \end{aligned} \tag{3.26}$$

$$\begin{aligned} F_-(\xi) = {} & \frac{i}{2\pi}\mathrm{P}\int_0^\infty F_+(\xi')\left[\;\ldots\; + \;\ldots\; - \;\ldots\; - \;\ldots\;\right]d\xi' \\ & + \frac{i}{2\pi}\int_0^\infty F_-(\xi')\left[\;\ldots\; - \;\ldots\; - \;\ldots\; + \;\ldots\;\right]d\xi'. \end{aligned} \tag{3.27}$$

Equations (3.23), (3.26) and (3.27) constitute a set of coupled singular integral equations to be solved for the functions $F_+(\xi)$ and $F_-(\xi)$ in the range $0 < \xi < \infty$. Equations (3.23) and (3.26) may be added to give a non-singular integral relation between $F_+(\xi)$ and $F_-(\xi)$, which together with equation (3.27) gives the system in its most compact form. Systems of singular integral equations are discussed in the book by Muskhelishvili (1946). For present purposes, however, we will merely assume that the system under discussion has a unique solution which represents the wave field.† Once $F_2(\xi)$ is determined, $F_1(\eta)$ and $F_3(\eta)$ are given by equations (3.2) for $\eta < 0$ and equations (3.3), (3.21) for $\eta > 0$.

In the special case where the surface is symmetric so that $\phi(\xi) = -\xi$ equations (3.24), (3.26) and (3.27) reduce to

$$\left.\begin{aligned} A(\xi) &= -\cos k_1\xi - \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{\sin k_1\xi' H_R'(\xi')\,d\xi'}{H_R(\xi')-H_R(\xi)}, \\ F_+(\xi) &= \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_-(\xi')\,2\xi'\,d\xi'}{\xi'^2-\xi^2}, \quad F_-(\xi) = \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_+(\xi')\,2\xi\,d\xi'}{\xi'^2-\xi^2}. \end{aligned}\right\} \tag{3.28}$$

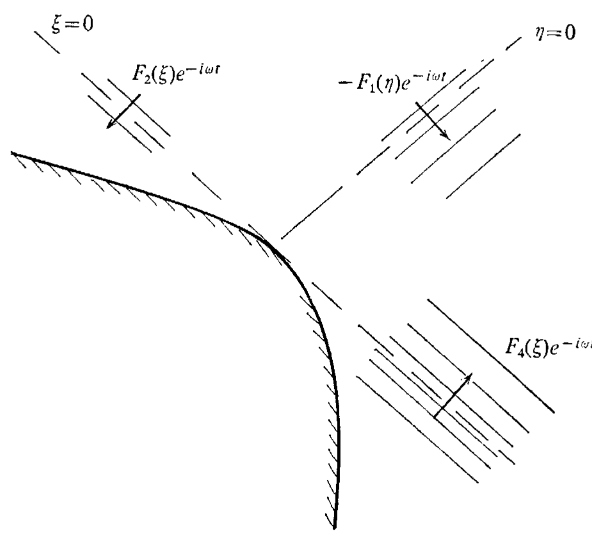

FIGURE 2. Diffraction topography with reflected and diffracted wave fields.

3.2. *The diffraction problem*

We next consider a plane wave incident on topography as shown in figure 2, with the origin again at the point where a characteristic is tangent to the surface. We denote the overall wave-field by a superposition of

$$\left.\begin{aligned} \psi_i &= \epsilon H_{ev}(\eta)\, e^{i(k_1\xi-\omega t)}, &&\text{the incident wave,} \\ \psi_R &= \epsilon F_2(\xi)\, e^{-i\omega t}, &&\text{the back-reflected wave,} \\ \psi_T &= -\epsilon F_1(\eta)\, e^{-i\omega t}, &&\text{the reflected wave,} \\ \psi_D &= \epsilon H_{ev}(-\eta)\, F_4(\xi)\, e^{-i\omega t}, &&\text{which contains the diffracted wave,} \end{aligned}\right\} \tag{3.29}$$

† It is not possible to determine the question of uniqueness from Mushkelishvili's work without specifying the functions $H_R(\xi)$ and $\phi(\xi)$. However, uniqueness seems intuitively plausible on physical grounds, and it is supported by the simple cases discussed below.

where $H_{ev}(\eta)$ denotes the Heaviside step function. The condition that ψ must vanish on the boundary then yields

$$\left.\begin{aligned} F_1(\eta) &= e^{ik_1\xi}+F_2(\xi), \quad \text{on} \quad \eta = -H(\xi) \quad (\eta > 0),\\ F_1(\eta) &= F_4(\xi), \quad \text{on} \quad \eta = -H(\xi) \quad (\eta < 0), \end{aligned}\right\} \quad (\xi > 0), \tag{3.30}$$

and we also have $$F_4(\xi) = e^{ik_1\xi}+F_2(\xi) \quad (\xi > 0). \tag{3.31}$$

Assuming that there are no other points on the surface which have grazing characteristics, and that the surface is asymptotic to flat planes at infinity, the statement of the problem is completed by the following radiation conditions:

$$F_2(\xi) = \frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{F_2(\xi')\,d\xi'}{\xi'-\xi}, \quad F_1(\eta) = \frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{F_1(\eta')\,d\eta'}{\eta'-\eta}, \quad F_4(\xi) = -\frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{F_4(\xi')\,d\xi'}{\xi'-\xi}.$$
(3.32), (3.33), (3.34)

The procedure for reducing this system to a soluble set of equations is analogous to that for the preceding case. Near the origin the equation for the surface has the form

$$-\xi = K(\eta) = \frac{(1+c^2)^{\frac{3}{2}}}{8c^2R}\eta^2(1+O(\eta/R)),$$

$$-\eta = H(\xi) = -\operatorname{sgn}\eta.\frac{2^{\frac{3}{2}}R^{\frac{1}{2}}c}{(1+c^2)^{\frac{3}{4}}}|\xi|^{\frac{1}{2}}(1+O(\xi/R)^{\frac{1}{2}}), \tag{3.35}$$

and $$H'(\xi) \sim \frac{2^{\frac{1}{2}}cR^{\frac{1}{2}}}{(1+c^2)^{\frac{3}{4}}|\xi|^{\frac{1}{2}}}, \tag{3.36}$$

where as before R is the radius of curvature at the origin. Again defining the left- and right-hand parts of the surface by

$$\left.\begin{aligned} \eta &= -H_R(\xi), \quad \xi = -K_R(\eta) \quad \text{for} \quad \eta < 0,\\ \eta &= -H_L(\xi), \quad \xi = -K_L(\eta) \quad \text{for} \quad \eta > 0, \end{aligned}\right\} \tag{3.37}$$

we define the function $\phi(\eta)$ by

$$\left.\begin{aligned} K_R(\phi(\eta)) &= K_L(\eta) \quad (\eta > 0),\\ K_L(\phi(\eta)) &= K_R(\eta) \quad (\eta < 0), \end{aligned}\right\} \tag{3.38}$$

and $\phi(\eta)$ clearly has the properties given by equation (3.12).

Next, we define

$$\left.\begin{aligned} G_+(\eta) &= \tfrac{1}{2}(F_1(\eta)+F_1(\phi(\eta))),\\ G_-(\eta) &= \tfrac{1}{2}(F_1(\eta)-F_1(\phi(\eta))), \end{aligned}\right\} \tag{3.39}$$

so that $$F_1(\eta) = G_+(\eta)+G_-(\eta), \quad F_1(\phi(\eta)) = G_+(\eta)-G_-(\eta). \tag{3.40}$$

Equations (3.30), (3.31), (3.32) yield

$$F_2(\xi) = \frac{i}{\pi}\int_0^{\infty}\frac{F_2(\xi')\,d\xi'}{\xi'-\xi}-\frac{i}{\pi}\int_0^{\infty}\frac{F_1(\eta')\,K_L'(\eta')\,d\eta'}{K_L(\eta')+\xi}-\frac{i}{\pi}\int_{-\infty}^{0}\frac{e^{ik_1\xi'}d\xi'}{\xi'-\xi} \quad (-\infty < \xi < \infty), \tag{3.41}$$

and (3.30), (3.31) and (3.33) similarly give

$$F_4(\xi) = -\frac{i}{\pi}\int_0^{\infty}\frac{F_2(\xi')\,d\xi'}{\xi'-\xi}+\frac{i}{\pi}\int_0^{\infty}\frac{F_1(\phi(\eta'))\,K_L'(\eta')\,d\eta'}{K_L(\eta')+\xi}-\frac{i}{\pi}\int_0^{\infty}\frac{e^{ik_1\xi'}d\xi'}{\xi'-\xi}$$
$$(-\infty < \xi < \infty), \tag{3.42}$$

where the second integral has been transformed in the same manner as equation (3.19). Adding (3.41) and (3.42) gives

$$F_2(\xi)+F_4(\xi) = -\frac{i}{\pi}\int_0^\infty \frac{[F_1(\eta')-F_1(\phi(\eta'))]\,K_L'(\eta')\,d\eta'}{K_L(\eta')-K_L(\eta)}+e^{ik_1\xi} \quad (-\infty<\xi<\infty), \tag{3.43}$$

using the relation

$$\frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ik_1\xi'}\,d\xi'}{\xi'-\xi} = -e^{ik_1\xi}. \tag{3.44}$$

Taking $\xi<0$, $\xi=-K_L(\eta)$, equation (3.43) gives

$$F_1(\eta)+F_1(\phi(\eta))-e^{-ik_1K_L(\eta)} = -\frac{i}{\pi}\int_0^\infty \frac{[F_1(\eta')-F_1(\phi(\eta'))]\,K_L'(\eta')\,d\eta'}{K_L(\eta')-K_L(\eta)}+e^{-ik_1K_L(\eta)} \quad (0<\eta<\infty), \tag{3.45}$$

and so

$$G_+(\eta) = -\frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{G_-(\eta')\,K_L'(\eta')\,d\eta'}{K_L(\eta')-K_L(\eta)}+e^{-ik_1K_L(\eta)} \quad (0<\eta<\infty). \tag{3.46}$$

Two other relations between $G_+(\eta)$ and $G_-(\eta)$ may be obtained from equation (3.33), and these are identical to equations (3.26), (3.27) with G_+, G_- replacing F_+, F_- and η, η' replacing ξ, ξ'. The system of equations for G_+, G_- is therefore identical to that for F_+, F_- in the previous case, with the exception of the inhomogeneous term. $F_2(\xi)$, $F_4(\xi)$ are given in terms of $F_1(\eta)$ by equations (3.30), (3.31), and (3.43) with $\xi>0$.

4. Properties of split reflexion

We aim to determine the general properties of the wave field on split reflexion by investigating the equations for some analytically simple cases. We consider first the simplest case of all, which is

(*a*) $$H(\xi) = c_1\xi^2 \quad (-\infty<\xi<\infty).$$

$c_1=(1+c^2)^{\frac{3}{2}}/8c^2R$, and $\phi(\xi)=-\xi$, for all ξ. The topography is illustrated in figure 3. Equation (3.23) becomes

$$F_+(\xi) = A(\xi)-\frac{i}{\pi}\mathrm{P}\int_0^\infty\frac{F_-(\xi')\,2\xi'\,d\xi'}{\xi'^2-\xi^2}, \tag{4.1}$$

and equations (3.28) give
$$A(\xi)=0, \tag{4.2}$$
so that
$$F_+(\xi)=0, \quad F_-(\xi)=0 \quad (0<\xi<\infty). \tag{4.3}$$
Hence
$$F_2(\xi)=0 \quad (-\infty<\xi<\infty), \tag{4.4}$$
and from equations (3.3), (3.21) for $\eta>0$ we have

$$F_1(\eta)=F_3(\eta) = -\frac{i}{\pi}\int_0^\infty\frac{\cos k_1\xi'\,.\,2\xi'\,d\xi'}{\xi'^2+\eta/c_1} = e^{-k_1(\eta/c_1)^{\frac{1}{2}}}, \tag{4.5}$$

upon evaluating the integral by conventional methods. The solution is completed by using equations (3.2), (3.9), and so we have

$$\left.\begin{aligned} F_1(\eta) &= e^{ik_1(|\eta|/c_1)^{\frac{1}{2}}}, \quad F_3(\eta) = e^{-ik_1(|\eta|/c_1)^{\frac{1}{2}}} \quad (-\infty<\eta<0),\\ &= e^{-k_1(\eta/c_1)^{\frac{1}{2}}}, \qquad\quad\;\; = e^{-k_1(\eta/c_1)^{\frac{1}{2}}} \quad (0<\eta<\infty),\\ F_2(\xi) &= 0, \quad (-\infty<\xi<\infty). \end{aligned}\right\} \tag{4.6}$$

This remarkably simple solution indicates what is probably the most significant feature of the present analysis, namely that the fluid velocity is proportional to $k_1 R^{\frac{1}{2}}/|\eta|^{\frac{1}{2}}$ near $\eta = 0$, so that the kinetic energy density is infinite in any volume enclosing part of this characteristic. The strength of the singularity in the

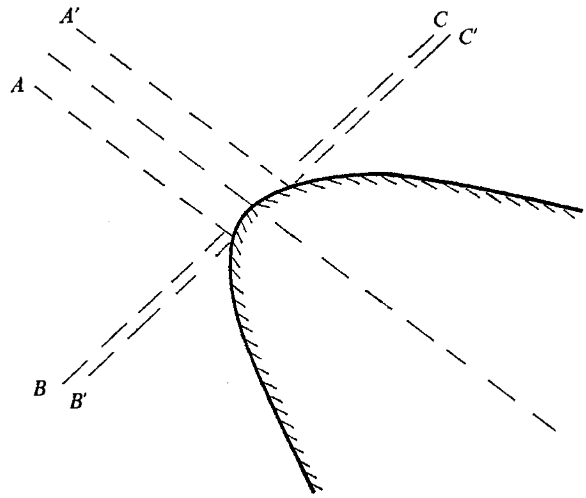

FIGURE 3. Split reflexion, the simplest case.

velocity is proportional to $k_1 R^{\frac{1}{2}}$ so that it is stronger for shorter incident wavelengths and larger radii of curvature. Furthermore, this singularity is not peculiar to this particular topography, as investigation of the governing equations shows that it occurs for virtually every type of topography which has the local behaviour given by equation (3.9). This is borne out by the more general example considered below. The presence of such a formidable singularity raises questions about the stability of such a system, and this has been discussed above. The vanishing of the back-reflected wave is a property of this particular surface shape: symmetric and parabolic in the characteristic co-ordinates, with slope at infinity asymptoting to that of the ξ-characteristics. It should also be noted that in the WKB limit of short incident wavelength ($k_1 R \gg 1$) for any smooth surface, the solution near the split-reflexion point must be asymptotic to the present one.

The reason for such a strong singularity becomes apparent if one considers energy fluxes near the critical point, as shown in figure 3. Since in this case there is no back reflexion, the energy flux incident between the lines A, A' (say) must be reflected between the lines B, B' and C, C' so that the incident energy is 'compressed' or 'squeezed' on reflexion and the energy density must be thereby increased. This squeezing is most severe near the grazing characteristic BC, and accounts for the infinite energy density. The large velocities near BC on the side $\eta > 0$ are rendered necessary by the dynamics of the system.

It is also worthwhile to interpret the process in terms of Fourier components. We consider how the field of motion is set up – the front of the incident wave (generated some large distance away) travels with its group velocity (Bretherton

1967) apart from some irrelevant forerunners, and on reaching the surface the incident energy is reflected in terms of a continuous spectrum of modes, many of which have higher wave-numbers. The front of each of these reflected modes will also travel with its appropriate group velocity, and for the high wave-number modes this is very small, so that the energy density associated with them accumulates near the point at the origin. The process of split reflexion generates these high wave-number modes in sufficient quantity so that, as the flow field approaches its final state, the energy density in the neighbourhood of the line $\eta = 0$ increases without limit.

For the second example we take

$$(b) \qquad \left.\begin{aligned} H(\xi) &= c_1\xi^2 \quad (\xi_L < \xi < \xi_R), \\ &\qquad (\xi_L < 0, |\xi_L| < \xi_R), \\ &= c_1\xi_R(2\xi-\xi_R) \quad (\xi > \xi_R), \\ &= c_1\xi_L(2\xi-\xi_L) \quad (\xi < \xi_L) \end{aligned}\right\} \tag{4.7}$$

so that $H(\xi)$ and its first derivative are continuous. The function $\phi(\xi)$ is then given by equations (3.11), which yield

$$\left.\begin{aligned} \phi(\xi) &= \frac{\xi_L}{\xi_R}\xi + \frac{1}{2\xi_R}(\xi_R^2-\xi_L^2), \quad -\infty < \xi < \frac{1}{2\xi_L}(\xi_R^2+\xi_L^2), \\ &= (2\xi\xi_L-\xi_L^2)^{\frac{1}{2}}, \quad \frac{1}{2\xi_L}(\xi_R^2+\xi_L^2) < \xi < \xi_L, \\ \phi(\xi) &= -\xi, \quad \xi_L < \xi < |\xi_L|, \\ &= \frac{1}{2\xi_L}(\xi^2+\xi_L^2), \quad |\xi_L| < \xi < \xi_R, \\ &= \frac{\xi_R}{\xi_L}\xi + \frac{1}{2\xi_L}(\xi_L^2-\xi_R^2), \quad \xi > \xi_R. \end{aligned}\right\} \tag{4.8}$$

We now investigate the nature of the solution near the origin and at large distances from it.

For $0 \leqslant \xi < |\xi_L|$, equations (3.26), (3.27), after some manipulation, may be written

$$F_+(\xi) = \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_-(\xi')\,2\xi'\,d\xi'}{\xi'^2-\xi^2} + a_0 + a_1 + (b_0+b_1)(k_1\xi)^2 + (d_0+d_1)(k_1\xi)^4 + \ldots, \tag{4.9}$$

$$F_-(\xi) = \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_+(\xi')\,2\xi'\,d\xi'}{\xi'^2-\xi^2} + (e_0+e_1)k_1\xi + (f_0+f_1)(k_1\xi)^3 + \ldots, \tag{4.10}$$

where $a_0, b_0, \ldots, f_0$ depend on the values of F_+, F_- between $|\xi_L|$ and ξ_R, and $a_1, b_1, \ldots, f_1$ depend on the values of F_+, F_- for $\xi > \xi_R$. Equation (3.23) gives

$$\begin{aligned} F_+(\xi) = A(\xi) - \frac{i}{\pi}\mathrm{P}\int_0^\infty \frac{F_-(\xi')\,2\xi'\,d\xi'}{\xi'^2-\xi^2} + \frac{i}{\pi}\int_{k_1\xi_R}^\infty F_-(\xi') \\ \times\left[\frac{2\xi'}{\xi'^2-(k_1\xi)^2} - \frac{1}{\xi'-(k_1/2\xi_R)(\xi^2+\xi_R^2)}\right]d\xi' \quad (0 \leqslant \xi < |\xi_L|), \end{aligned} \tag{4.11}$$

so that adding to (4.9) gives

$$F_+(\xi) = \tfrac{1}{2}A(\xi)+\tfrac{1}{2}a_0+a_1'+(\tfrac{1}{2}d_0+d_1')(k_1\xi)^2+(\tfrac{1}{2}d_0+d_1')(k_1\xi)^4+\ldots \quad (0 \leqslant \xi < |\xi_L|), \tag{4.12}$$

where a_0, a_1' etc. are constants.

From equation (3.24) we have, for $0 \leqslant \xi < |\xi_L|$,

$$\begin{aligned} A(\xi) &= \frac{i}{2\pi}\int_{|\xi_L|}^{\xi_R} \frac{(e^{ik_1\xi_L/2}e^{ik_1\xi'^2/2\xi_L}-e^{ik_1\xi'})}{\xi'^2-\xi^2}2\xi'\,d\xi' \\ &\quad+\frac{i}{2\pi}\int_{|\xi_L|}^{\infty}\frac{\sin k_1\xi'\,.\,2\xi'\,d\xi'}{\xi'^2-\xi^2} \\ &\quad-\frac{i}{2\pi}\int_{\xi_R}^{\infty}\frac{(e^{ik_1\xi'}-e^{ik_1\xi_R\xi'/\xi_L}e^{(ik_1/2\xi_L)(\xi_L^2-\xi_R^2)})}{2\xi_R\xi'-\xi^2-\xi_R^2}2\xi_R\,d\xi', \end{aligned} \tag{4.13}$$

$$= A_0+A_1(k_1\xi)^2+A_2(k_1\xi)^4+\ldots, \tag{4.14}$$

where

$$\begin{aligned} A_0 &= \frac{i}{\pi}\int_{|\xi_L|}^{\xi_R}\frac{(e^{ik_1\xi_L/2}e^{ik_1\xi'^2/2\xi_L}-e^{ik_1\xi'})}{\xi'}\,d\xi'+\frac{i}{\pi}\int_{|\xi_L|}^{\xi_R}\frac{\sin k_1\xi'}{\xi'}\,d\xi' \\ &\quad+\frac{i}{\pi}\int_{\xi_R}^{\infty}\left[\frac{\sin k_1\xi'}{\xi'}-\frac{\xi_R(e^{ik_1\xi'}-e^{ik_1\xi_R\xi'/\xi_L}e^{(ik_1/2\xi_L)(\xi_L^2-\xi_R^2)})}{2\xi_R\xi'-\xi_R^2}\right]d\xi', \end{aligned} \tag{4.15}$$

etc.

Hence we see that $F_+(\xi)$ is necessarily analytic near $\xi = 0$, and substituting in equation (4.10) shows (after proper consideration of the principal value singularity) that $F_-(\xi)$ is analytic there also.† Hence the back-reflected wave $F_2(\xi)$ will in general be present near $\xi = 0$ owing to the position and slope of the distant parts of the surface, and the velocities associated with it will be finite and continuous there. Furthermore, from the nature of the integrals above on which these properties depend, it is clear that they extend to most surfaces in general, a possible exception being those surfaces which have singularities in their higher derivatives at the origin. (For the example given here, the reflected waves would be expected to have discontinuities and singularities in their velocity gradients near the points (ξ_L, η_L), (ξ_R, η_R), since the surface has discontinuous second derivatives there.) Since the back-reflected wave is necessarily analytic near $\xi = 0$, the $F_1(\eta)$, $F_3(\eta)$ waves will still have the $(k_1c_1|\eta|)^{\frac{1}{2}}$ singularity at the origin, as in example (*a*).

For $\xi \gg \xi_R$, $A(\xi)$ may be written in the form

$$\begin{aligned} A(\xi) &= \frac{i}{2\pi}e^{ik_1\xi}\int_{-\infty}^{k_1(\xi_R-\xi)}\frac{e^{iv}\,dv}{v}+\frac{i}{2\pi}e^{(ik_1\xi_R\xi/\xi_L)}e^{(ik_1/2\xi_L)(\xi_L^2-\xi_R^2)}\int_{-\infty}^{(k_1\xi_R/\xi_L)(\xi_R-\xi)}\frac{e^{iv}\,dv}{v} \\ &\quad-\frac{i}{2\pi}\int_0^{|\xi_L|}\frac{\sin k_1\xi'\,.\,2\xi'\,d\xi'}{\xi'^2-\xi^2}-\frac{i}{2\pi}\int_{|\xi_L|}^{\xi_R}\frac{(e^{ik_1\xi'}-e^{(ik_1/2\xi_L)(\xi'^2+\xi_L^2)})2\xi'\,d\xi'}{\xi'^2-\xi^2}, \end{aligned} \tag{4.16}$$

$$= -\frac{e^{ik_1\xi_R}}{2\pi k_1(\xi-\xi_R)}-\frac{\xi_L\,e^{(ik_1/2\xi_L)(\xi_L^2+\xi_R^2)}}{2\pi k_1\xi_R(\xi-\xi_R)}+O\left(\frac{1}{\xi^2}\right). \tag{4.17}$$

† This property is dependent on the fact that the expansion for $F_+(\xi)$ about the origin contains only even powers of ξ.

A similar argument to that given above shows that, for ξ large, $F_+(\xi)$ and $F_-(\xi)$ are $O(1/k_1(\xi-\xi_R))$, so that

$$F_2(\xi) = O\left(\frac{1}{k_1(\xi-\xi_R)}\right) \quad (|\xi| \gg \xi_R), \tag{4.18}$$

for the example considered. Consequently the reflected waves will asymptote to plane waves as expected.

5. Properties of diffraction

As for the split reflexion case, we investigate some simple topographies and infer the general properties of diffraction of internal/inertial waves from these. Again, there is one case which is particularly simple, namely

(*a*) $$K(\eta) = c_1\eta^2 \quad (-\infty < \eta < \infty).$$

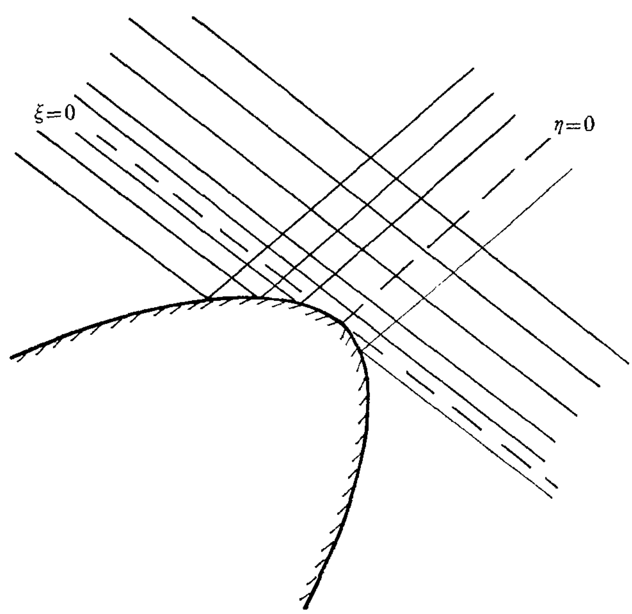

FIGURE 4. Diffraction, the simplest case.

This topography is illustrated in figure 4, with

$$c_1 = (1+c^2)^{\frac{3}{2}}/8c^2R, \quad \text{and} \quad \phi(\eta) = -\eta,$$

for all η. Equation (3.46) becomes

$$G_+(\eta) = e^{-ik_1c_1\eta^2} - \frac{i}{\pi}\,\mathrm{P}\int_0^\infty \frac{G_-(\eta')\,2\eta' d\eta'}{\eta'^2-\eta^2} \quad (0 < \eta < \infty), \tag{5.1}$$

and the equations corresponding to (3.26), (3.27) are

$$G_+(\eta) = \frac{i}{\pi}\,\mathrm{P}\int_0^\infty \frac{G_-(\eta')\,2\eta' d\eta'}{\eta'^2-\eta^2}, \quad G_-(\eta) = \frac{i}{\pi}\,\mathrm{P}\int_0^\infty \frac{G_+(\eta')\,2\eta\, d\eta'}{\eta'^2-\eta^2} \quad (0 < \eta < \infty). \tag{5.2}$$

Adding the first of these to (5.1) gives

$$G_+(\eta) = \tfrac{1}{2}e^{-ik_1c_1\eta^2} \quad (0 < \eta < \infty), \tag{5.3}$$

so that

$$G_-(\eta) = \frac{i}{2\pi}\,\mathrm{P}\int_0^\infty \frac{e^{-ik_1c_1\eta^2}\,2\eta\,d\eta'}{\eta'^2-\eta^2} \quad (0 < \eta < \infty). \tag{5.4}$$

This latter integral may be written

$$G_-(\eta) = -\frac{1}{2\pi i}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{-ik_1c_1\eta'^2}\,d\eta'}{\eta'-\eta}, \tag{5.5}$$

$$= -\lim_{\delta\to 0+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-ik_1c_1\eta'^2}\,d\eta'}{\eta'-(\eta+i\delta)}+\tfrac{1}{2}e^{-ik_1c_1\eta^2}, \tag{5.6}$$

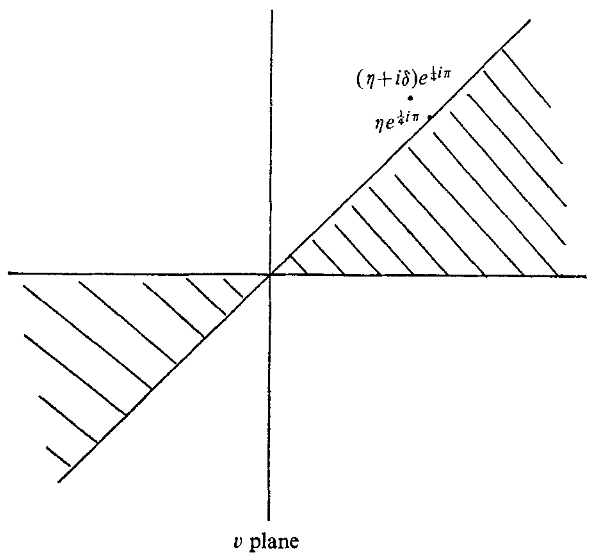

FIGURE 5. The contour for equation (5.8).

by the Plemelj formulae of complex analysis, and writing

$$v = e^{\frac{1}{4}i\pi}(k_1c_1)^{\frac{1}{2}}\,\eta' \tag{5.7}$$

we have

$$G_-(\eta) = \tfrac{1}{2}\,e^{-ik_1c_1\eta^2} - \lim_{\delta\to 0+}\frac{1}{2\pi i}\int_{-(1+i)\infty}^{(1+i)\infty}\frac{e^{-v^2}\,dv}{v-e^{\frac{1}{4}i\pi}(k_1c_1)^{\frac{1}{2}}(\eta+i\delta)}, \tag{5.8}$$

and since the integrand approaches zero as $|v|\to\infty$ in the shaded region in figure 5 and the pole lies outside it, we have

$$G_-(\eta) = \tfrac{1}{2}\,e^{-ik_1c_1\eta^2} - \lim_{\delta\to 0+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-v^2}\,dv}{v-e^{\frac{1}{4}i\pi}(k_1c_1)^{\frac{1}{2}}(\eta+i\delta)}, \tag{5.9}$$

$$= \frac{i}{\pi^{\frac{1}{2}}}\,e^{-ik_1c_1\eta^2}\int_0^{\exp\{\frac{1}{4}i\pi(k_1c_1)^{\frac{1}{2}}\eta\}} e^{t^2}\,dt, \tag{5.10}$$

using a standard result for complementary error functions (Abramovitz &

Stegun 1965 p. 297, hereafter denoted A & S). Writing $t = (\frac{1}{2}\pi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} t'$, this becomes

$$G_-(\eta) = \frac{e^{-\frac{1}{4}i\pi} \cdot e^{-ik_1c_1\eta^2}}{2^{\frac{1}{2}}} \int_0^{(2k_1c_1/\pi)^{\frac{1}{2}}\eta} e^{\frac{1}{2}i\pi t'^2}\, dt' \tag{5.11}$$

$$= \frac{e^{-\frac{1}{4}i\pi} \cdot e^{-ik_1c_1\eta^2}}{2^{\frac{1}{2}}} \left[C\left(\left(\frac{2k_1c_1}{\pi} \right)^{\frac{1}{2}} \eta \right) + iS\left(\left(\frac{2k_1c_1}{\pi} \right)^{\frac{1}{2}} \eta \right) \right], \tag{5.12}$$

where $C(x)$ and $S(x)$ are the Fresnel integrals, as defined in A & S (p. 300). Hence, from equations (3.39), we have

$$F_1(\eta) = \frac{e^{-ik_1c_1\eta^2}}{2} \left[1 + e^{-\frac{1}{4}i\pi}\, 2^{\frac{1}{2}} \left(C\left(\left(\frac{2k_1c_1}{\pi} \right)^{\frac{1}{2}} \eta \right) + iS\left(\left(\frac{2k_1c_1}{\pi} \right)^{\frac{1}{2}} \eta \right) \right) \right], \tag{5.13}$$

and this is valid for all η. For $\xi < 0$, $F_2(\xi)$, $F_4(\xi)$ are given by equations (3.30), which yield

$$F_2(\xi) = -\frac{e^{ik_1\xi}}{2} \left[1 - e^{-\frac{1}{4}i\pi}\, 2^{\frac{1}{2}} \left(C\left(\left(\frac{2k_1}{\pi} |\xi| \right)^{\frac{1}{2}} \right) + iS\left(\left(\frac{2k_1}{\pi} |\xi| \right)^{\frac{1}{2}} \right) \right) \right], \tag{5.14}$$

$$F_4(\xi) = +\frac{e^{ik_1\xi}}{2} \left[1 - \qquad \cdots \qquad \cdots \qquad \right]. \tag{5.15}$$

For $\xi > 0$ we have equations (3.31), (3.43), which give

$$F_2(\xi) = \frac{1}{\pi i} \int_0^\infty \frac{G_-(\eta')\, 2\eta'\, d\eta'}{\eta'^2 + \xi/c_1} \quad (0 < \xi < \infty), \tag{5.16}$$

$$F_4(\xi) = \qquad \cdots \qquad + e^{ik_1\xi}, \tag{5.17}$$

with $G_-(\eta)$ given by (5.12). Changing the variable of integration, equation (5.16) may be written

$$F_2(\xi) = \frac{e^{-\frac{1}{4}i\pi}}{2^{\frac{1}{2}}\pi i} \int_0^\infty \frac{e^{-\frac{1}{2}i\pi u^2} (C(u) + iS(u))\, 2u\, du}{u^2 + 2k_1\xi/\pi} \quad (0 < \xi < \infty), \tag{5.18}$$

$$\left.\begin{aligned} &\sim -\tfrac{1}{2} - \frac{e^{-\frac{1}{4}i\pi} (k_1\xi)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \quad (k_1\xi \text{ small}), \\ &= O\left(\frac{1}{k_1\xi}\right) \quad (k_1\xi \text{ large}). \end{aligned}\right\} \tag{5.19}$$

Equations (5.13)–(5.18) constitute the complete solution to the wave field, and $C(x)$, $S(x)$ are illustrated in figure 6. First we may note that the back-reflected wave $F_2(\xi)$ and the 'diffracted' wave $F_4(\xi)$ are independent of the radius of curvature R, so that the form of these waves does not depend on whether the surface is broad or thin. Second, near the line $\xi = 0$ both $F_2(\xi)$ and $F_4(\xi)$ behave like $k_1|\xi|^{\frac{1}{2}}$ on each side, so that the singularity in the velocity and energy density obtained for split reflexion is present here also, but with its strength dependent only on the incident wave-number k_1. From equation (5.15) we see that in the geometric shadow region the diffracted wave only penetrates a distance ξ_d of the order of

$$\xi_d = 2\pi/k_1, \tag{5.20}$$

i.e. one wavelength of the incident wave (A & S, p. 301), and has the form of a plane wave travelling in the same direction as the incident wave, modulated by a complex term involving Fresnel integrals. The corresponding part of the reflected wave $F_1(\eta)$, $\eta < 0$, is only significant near $\eta = 0$ with a decay distance of

$$\eta_d = \frac{4\pi^{\frac{1}{2}}c}{(1+c^2)^{\frac{3}{4}}}\left(\frac{R}{k_1}\right)^{\frac{1}{2}}. \tag{5.21}$$

$F_1(\eta)$ has no singular behaviour. As in §4, the above solution will apply for any smooth diffracting surface in the limit $k_1 R \gg 1$, so that these results have some generality.

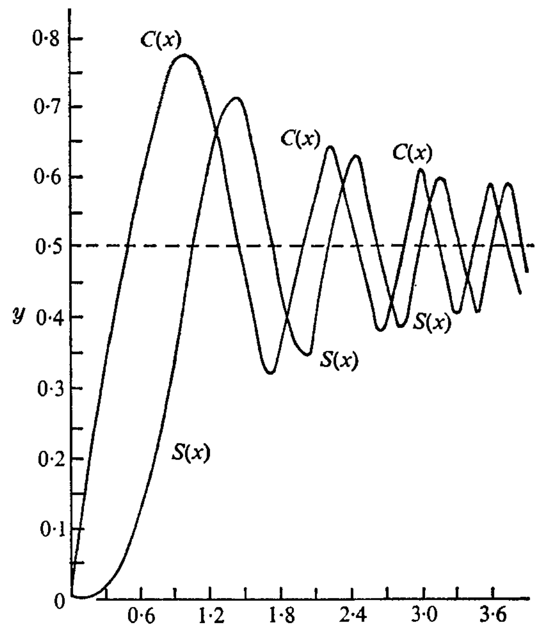

FIGURE 6. The Fresnel integrals $C(x)$ and $S(x)$, from Abramovitz & Stegun (1965, p. 301).

We may consider the case analogous to case (*b*) of the preceding section, which is to take

$$\left.\begin{aligned} K(\eta) &= c_1\eta^2 && (\eta_L < \eta < \eta_R), \\ &= c_1\eta_R(2\eta-\eta_R) && (\eta < \eta_R), \\ &= c_1\eta_L(2\eta-\eta_L) && (\eta > \eta_L). \end{aligned}\right\} \tag{5.22}$$

Precisely the same argument may be made, *mutatis mutandis*, and conclusions drawn concerning the way in which changes in the 'distant' topography may alter $F_1(\eta)$, $F_2(\xi)$, and $F_4(\xi)$ near the origin. The only change is that the function $A(\xi)$ is replaced by $e^{-ik_1K(\eta)}$, which equals $e^{-ik_1c_1\eta^2}$ near the origin. The specific conclusions for this case are that, regardless of the slope of the distant topography, near the origin $F_1(\eta)$ will be analytic and $F_2(\xi)$, $F_4(\xi)$ still have the $1/|\xi|^{\frac{1}{2}}$ singularity in the velocity and consequent logarithmic singularity in the energy density.

6. Asymptotic behaviour of reflected and diffracted wave fields

If we consider a rigid surface whose slope varies in an arbitrary manner in some finite region, but outside of which the surface is effectively plane on either side, we may obtain the asymptotic form for the wave field far from the region (i.e. far from characteristics which pass through the region) from the radiation condition, equation (2.12), alone. We consider topography as shown in figure 7 (for example), and for the back-reflected wave we have

$$F_2(\xi) = \frac{i}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi}. \tag{6.1}$$

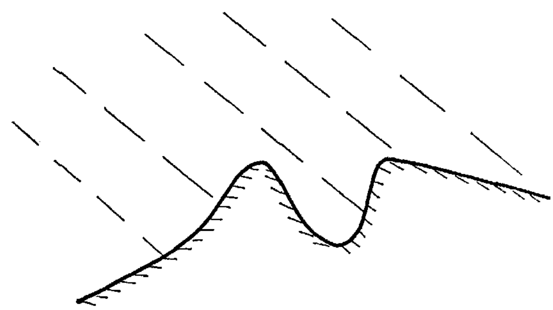

FIGURE 7. An example of localized topography.

We suppose that $F_2(\xi)$ is only $O(1)$ for $X_1 < \xi < X_2$ when X_1, X_2, are fixed for given topography, and that

$$F_2(\xi) \to 0 \quad \text{as} \quad \xi \to \infty.$$

Then for suitable fixed values of $\alpha > 0$, $\beta > 1$, and $|\xi|$ large so that

$$-\beta|\xi| < X_1 < X_2 < \xi - \alpha < \xi + \alpha < \beta|\xi|, \tag{6.2}$$

equation (6.1) may be written

$$\begin{aligned} F_2(\xi) = {} & \frac{i}{\pi} \int_{X_1}^{X_2} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi} + \frac{i}{\pi} \mathrm{P} \int_{\xi-\alpha}^{\xi+\alpha} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi} \\ & + \frac{i}{\pi} \int_{-\infty}^{-\beta|\xi|} + \int_{\beta|\xi|}^{\infty} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi} + \frac{i}{\pi} \int_{-\beta|\xi|}^{X_1} + \int_{X_2}^{\xi-\alpha} + \int_{\xi+\alpha}^{\beta|\xi|} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi}, \end{aligned} \tag{6.3}$$

with an analogous expression for the case when $\xi < X_i$.

The first term gives the contribution to the integral from the region where $F_2(\xi)$ is largest, and we make the additional, plausible assumption that this term contributes to the leading term for $F_2(\xi)$ when $|\xi|$ is large. For $\xi \gg X_2$ or $\xi \ll X_1$,

$$\int_{X_1}^{X_2} \frac{F_2(\xi')\, d\xi'}{\xi' - \xi} = -\int_{X_1}^{X_2} F_2(\xi')\, d\xi' \cdot \frac{1}{\xi}\left(1 + O\left(\frac{1}{\xi}\right)\right), \tag{6.4}$$

so that this term indicates that

$$F_2(\xi) \sim iA/\pi\xi \quad \text{as} \quad |\xi| \to \infty, \tag{6.5}$$

where A is some constant. We must verify that this expression is consistent with equation (6.1) itself. Substituting this form for $F_2(\xi)$ in the ranges of integration outside (X_1, X_2) gives, to the leading orders for each,

$$F_2(\xi)-\frac{i}{\pi}\int_{X_1}^{X_2}\frac{F_2(\xi')\,d\xi'}{\xi'-\xi}=\frac{iA}{\pi}\left\{-\frac{2\alpha}{\xi^2}+\frac{1}{\xi}\log\left[\frac{\beta+\xi/|\xi|}{\beta-\xi/|\xi|}\right]\right.$$

$$\left.+\frac{1}{\xi}\left(\log\left(\frac{\beta-\xi/|\xi|}{\beta+\xi/|\xi|}\right)+\log\left(\frac{1+\alpha/\xi}{1-\alpha/\xi}\right)+\log\left|\frac{X_2}{X_1}\right|+\log\left(\frac{1+|X_1|/\xi}{1+|X_2|/\xi}\right)\right)\right\}, \quad (6.6)$$

$$=\frac{iA}{\pi\xi}\log\left|\frac{X_2}{X_1}\right|+O(1/\xi^2). \quad (6.7)$$

Hence the above asymptotic form for $F_2(\xi)$ is self-consistent. If we choose the co-ordinate system so that the origin is situated near the centre of the bottom variations, we may take $X_1 = -X_2$, so that

$$A \approx \int_{-X_2}^{X_2} F_2(\xi')\,d\xi'. \quad (6.8)$$

If the wave field approaches a plane wave on either side rather than zero, (as is the case for the 'transmitted' wave $F_1(\eta)$ in figure 7), the asymptotic form is

$$F_2(\xi) \sim \text{plane wave}+\frac{iA}{\pi\xi}+O\left(\frac{1}{\xi^2}\right), \quad (6.9)$$

as may readily be seen by considering integrals like

$$\int_X^\infty \frac{e^{ik\xi'}\,d\xi'}{\xi'+\xi}=O\left(\frac{1}{k(X+\xi)}\right). \quad (6.10)$$

The above arguments are applicable for the various wave fields involved in split reflexion and diffraction, as in case (b) for §§4 and 5, for example (they do not, however, apply to case (a) since the surfaces there do not asymptote to planes). The velocity field decays as $1/\xi^2$ and the energy density as $1/\xi^4$.

This work has been supported by the following grants: NSF Grant GA 402 X3, Nonr-1841(74) NR 083-157, and N00014-67-A-0204-0048.

REFERENCES

ABRAMOVITZ, M. & STEGUN, I. 1965 *Handbook of Mathematical Functions*. Dover.

BAINES, P. G. 1971 The reflexion of internal/inertial waves from bumpy surfaces. *J. Fluid Mech.* **46**, 273.

BRETHERTON, F. P. 1967 The time-dependent motion due to a cylinder moving in an unbounded rotating or stratified fluid. *J. Fluid Mech.* **28**, 545.

CACCHIONE, D. 1970 Ph.D. Thesis, Massachusetts Institute of Technology.

HURLEY, D. G. 1970 Internal waves in a wedge-shaped region. *J. Fluid Mech.* **43**, 97.

MUSKHELISHVILI, N. 1946 *Singular Integral Equations*. Gröningen: P. Noordhoff.

PHILLIPS, O. M. 1963 Energy transfer in rotating fluids by reflexion of inertial waves. *Phys. Fluids*, **6**, 513.

ROSENFELD, L. 1961 Questions of irreversibility and ergodicity. In *Ergodic Theories*. Academic.